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Case Study

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Application of the Castigliano's Second Theorem in the Solution of Statically Indeterminate Plane Structures

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Abstract

The paper present is to analyse statically indeterminate plane structures, using the Castigliano's second theorem and the "Theorem of Last Work" [1] in the determination of the statically indeterminate constants. For this, we consider models of structures subjected to different loads where the boundary conditions in the external supports are defined, the statically indeterminate constants being determined as unknowns. Noteworthy are the analysis of the classical equations presented for Timoshenko S.P. e Young D.H. (1965) [1] of the elastic theories in the development of the mathematical models. We use numerical simulation using the finite element program (FEA) Ansys® [2], where we compare the numerical and analytical results obtained by the equations of the present work.

Keywords: Structures, Castigliano's second theorem, Finite Elements.

Introduction

In 1879 Alberto Castigliano, an Italian railroad engineer, published a book in which he outlined a method for determining the deflection or slope at a given point in a structure, be it a truss, beam, or frame. This method, which is referred to as Castigliano's second theorem, or the method of last work, applies only to structures that have constant temperature, unyielding supports, and linear elastic material response. If the displacement of a point is to be determined, the theorem states that it is equal to the first partial derivative of the strain energy in the structure with respect to force acting at a point and in the direction of displacement. In a similar manner, the slope at a point in a structure is equal to the first partial derivative of the strain energy in the structure with respect to a couple moments acting at the point and in the direction of rotation. Among the works on the application of the Castigliano's second theorem, the classic theories of Timoshenko S.P. and Young D.H. (1965) [1] and the works of Hibbeler R.C. (2013) [3], among others [4][5][6], stand out.

Castigliano's Second Theorem

When an elastic material undergoes deformations, the transformation of the external work occurs due to the external loads in strain energy. This energy is accumulated in the material during deformation, which after release causes the material to return to its original state of rest. To derive Castigliano's second theorem, consider a structure of any arbitrary shape which is subjected to a series of n forces $P_1, P_2, ..., P_n$. Since the external work done by these loads is equal to the internal strain energy stored in the structure, we can write,

$$U_i = U_e \qquad (1)$$

The external work is a function of external loads. Thus,

$$U_i = U_e = f(P_1, P_2, ..., P_n) \qquad (2)$$

Now, if any of the forces, says Pi, is increased by a differential amount dPi, the internal work is also increased such that the new strain energy becomes,

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_i} dP_i \qquad (3)$$

This value, however, should not depend on the sequence in which the n forces are applied to the structure. For example, if we first apply dP_i to the structure first, then this will cause the structure to be displaced a differential amount $d\Delta_i$ in the direction of dP_i. If $U_e = ½ P\Delta$, the increment of strain energy would be ½ $dP_i d\Delta_i$. This quantity, however, is a second-order differential and may be neglected, as adopted in the present work. Further application of the loads $P_1, P_2, ..., P_n$, which displace the structure $\Delta_1, \Delta_2, ..., \Delta_n$, yields the strain energy.

$$U_i + dU_i = U_i + dP_i \Delta_I \quad (4)$$

Here, as before, *Ui* is the internal strain energy in the structure, caused by the loads P_1, P_2, ..., P_n, and $dP_i d\Delta_I$ am the additional strain energy caused by dP_i.

In summary, then, Eq. (3) represents the strain energy in the structure determined by first applyingthe loads P_1, P_2, ..., P_n, then dP_i, and Eq. (4) represents the strain energy determined by first applying dP_i and then the loads P_1, P_2, ..., P_n. Since these two equations must be equal, we require,

$$\Delta_i = \frac{\partial U_i}{\partial P_i} \quad (5)$$

Which proves the theorem; i.e., the displacement Δ_i in the direction of P_i am equal to the first partialderivative of the strain energy with respect to P_i.

It should be noted that Eq (5) is a statement regarding the structure's compatibility. Also, the above derivation requires that only conservative forces be considered for the analysis. These forces do work that is independent of the path and therefore crate no energy loss. Since forces causing a linear elasticresponse are conservative, the theorem is restricted to linear elastic behavior of the material.

Case Study

In the respective case studies, the equation of the Castigliano's second theorem will be used to calculate the flexibility coefficients. The symbology adopted is described below:

δ_{i0} - Displacement at the point where the load is applied X_i due to the actual load of the structure,"*i*" varies from 1 to the total number of constants;
δ_{ii} - Displacement at the point where the load X_i is applied due to the load X_i;δ_{ij} - Displacement at the point where the load X_i is applied due to load X_j;
$\delta_{ij} = \delta_{ji}$ - reciprocity of the displacements;
k_i – Spring constant elastic;
X'_i - Virtual load at the point of application of charge X_i;
n - Number of sections for the determination of internal efforts;
m - Number of span or segments of the structure;
M_n – Internal bending moment in section "*n*";
N_n – Internal normal force at section "*n*"; T_n - Internal torsion moment at section "*n*";Q_n - Internal shear force at section "*n*";

From the cross section and material by span or segment of the structure, we have:
I_m - Moments of inertia;
A_m - Area;
J_m – Polar moments of inertia;
A_{vm} - Effective area of shear;
E - Modulus of elasticity for the material;
G – Shear modulus of elasticity for the material;
EI_m - Flexural stiffness;
EA_m - Normal effort rigidity;GJ_m - Torsion stiffness; GA_{vm} - Shear stiffness;
L - Interval of integration or length of the structure segment.

The main equations used in the calculation of the coefficients of flexibility: Coefficient of flexibility at the point where the load "*i*" is applied due to the actual load:

$$\delta_{i0} = \sum_1^n \left(\int_L \frac{M_n}{EI_m} \frac{\partial M_n}{\partial X'_i} dx_n + \int_L \frac{N_n}{EA_m} \frac{\partial N_n}{\partial X'_i} dx_n + \int_L \frac{T_n}{GJ_m} \frac{\partial T_n}{\partial X'_i} dx_n + \int_L \frac{Q_n}{GAv_m} \frac{\partial Q_n}{\partial X'_i} dx_n \right) \quad (6)$$

Coefficient of flexibility at the point where the load "*i*" is applied due to the load "*i*" itself:

$$\delta_{ii} = \frac{1}{k_i} + \sum_1^n \left[\int_L \frac{1}{EI_m} \left(\frac{\partial M_n}{\partial X'_i} \right)^2 dx_n + \int_L \frac{1}{EA_m} \left(\frac{\partial N_n}{\partial X'_i} \right)^2 dx_n + \int_L \frac{1}{GJ_m} \left(\frac{\partial T_n}{\partial X'_i} \right)^2 dx_n + \int_L \frac{1}{GAv_m} \left(\frac{\partial Q_n}{\partial X'_i} \right)^2 dx_n \right]$$

Flexibility coefficient at the point where load "*i*" is applied due to load "*j*":

$$\delta_{ij} = \sum_1^n \left(\int_L \frac{1}{EI_m} \frac{\partial M_n}{\partial X'_i} \frac{\partial M_n}{\partial X'_j} dx_n + \int_L \frac{1}{EA_m} \frac{\partial N_n}{\partial X'_i} \frac{\partial N_n}{\partial X'_j} dx_n + \int_L \frac{1}{GJ_m} \frac{\partial T_n}{\partial X'_i} \frac{\partial T_n}{\partial X'_j} dx_n + \int_L \frac{1}{GAv_m} \frac{\partial Q_n}{\partial X'_i} \frac{\partial Q_n}{\partial X'_j} dx_n \right)$$

Since, after the respective partial derivatives, the charges when defined as virtual X'_i are equal tozero, otherwise it will be equal to the value of the actual load applied.

After determining the flexibility coefficients, the displacement compatibility equation must be applied to obtain the statically indeterminate constants. In specific cases, where there is no elastic support but only rigid supports, where the respective displacements are equal to zero, the Theorem ofLast Work can be applied for the determination of the statically indeterminate constants. Therefore, the equations of compatibility of displacements and for the Theorem of Last Work are given, respectively, by:

$$\begin{aligned} &\delta_{10} + \delta_{11} X_1 + \delta_{12} X_2 + \ldots + \delta_{1\eta} X_\eta = 0 \\ &\delta_{20} + \delta_{21} X_1 + \delta_{22} X_2 + \ldots + \delta_{2\eta} X_\eta = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\delta_{\eta 0} + \delta_{\eta 1} X_1 + \delta_{\eta 2} X_2 + \ldots + \delta_{\eta\eta} X_\eta = 0 \end{aligned} \quad (9)$$

$$\delta_{i0} = 0 = \sum_1^n \left(\int_L \frac{M_n}{EI_m} \frac{\partial M_n}{\partial X'_i} dx_n + \int_L \frac{N_n}{EA_m} \frac{\partial N_n}{\partial X'_i} dx_n + \int_L \frac{T_n}{GJ_m} \frac{\partial T_n}{\partial X'_i} dx_n + \int_L \frac{Q_n}{GAv_m} \frac{\partial Q_n}{\partial X'_i} dx_n \right) \quad (10)$$

In the work developed by Timoshenko S.P. and Young DH (1965) [1], Hibbeler R.C. (2013) [3], Soriano H.L. and S.S. Lima (2006) [4], among others, it is verified that the influence of shear forces represents approximately 5% in many cases can be disregarded, representing a significant simplification in the calculations. In the case study, presented in item 3.1, it is shown the percentage variation of the influence of the shear forces on the final result. In the other case studies, this influence is not considered as a function of the analysis presented in the aforementioned item.

Statically indeterminate beam supported on the two springs

Consider the statically indeterminate beam, set at one end and supported on the two springs of elasticconstants k_1 = 20000 kN/m and k_2 = 30000 kN/m, applied to supports "*B*" and "*C*" respectively. The beam has a constant bending stiffness *EI* = 18370,8 kNm² and it is subjected to uniformly distributed loads q_1 = 2,0 kN/m and q_2 = 4,0 kN/m, a concentrated load *P* = 10,0 kN and at a concentrated moment *M* = 20,0 kNm. The beam has dimensions L_1 = 3,0 m, L_2 = 2,0 m and L_3 = 4,0 m, as represented in Figure 1 below.

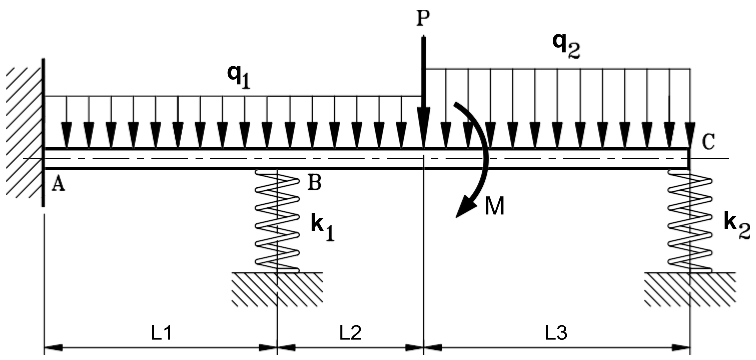

Figure 1: Statically indeterminate beam.

The representation of the adopted cuts, as well as the definition of the statically indeterminateconstants, are represented in Figure 2 below.

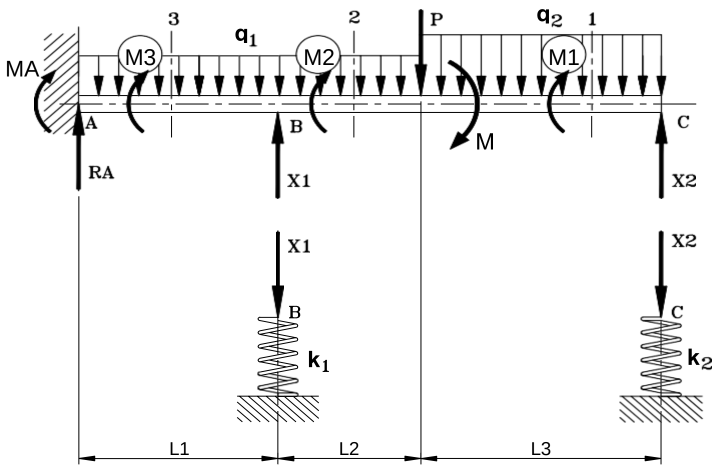

Figure 2: Sections and unknowns constants.

For the determination of these forces, bending moment and constant forces, we consider the sections 1, 2 and 3, defined according to the distribution of the loads and supports. Supports "*B*" and "*C*" will be defined as superabundant, with X_1 and X_2 being the unknown's constants. In this step, we consider X'_i (*i* = 1 and 2) as virtual loads.

Section 1: $0 \leq x_1 \leq L_3$

$$M_1 = -\frac{q_2}{2}x_1^2 + X'_2\, x_1 \tag{11.a}$$

$$Q_1 = \frac{\partial M_1}{\partial x_1} = -q_2 x_1 + X'_2 \tag{11.b}$$

Section 2: $L_3 \leq x_2 \leq (L_2+L_3)$

$$M_2 = -\frac{q_1}{2}(x_2 - L_3)^2 - q_2 L_3\left(x_2 - \frac{L_3}{2}\right) - M - P(x_2 - L_3) + X'_2\, x_2 \tag{12.a}$$

$$Q_2 = \frac{\partial M_2}{\partial x_2} = -q_1(x_2 - L_3) - q_2 L_3 - P + X'_2 \tag{12.b}$$

Section 3: $(L_2+L_3) \leq x_3 \leq (L_1+L_2+L_3)$

$$M_3 = -\frac{q_1}{2}(x_3 - L_3)^2 - q_2 L_3\left(x_3 - \frac{L_3}{2}\right) - M - P(x_3 - L_3) + X'_1\left[x_3 - (L_2 + L_3)\right] + X'_2\, x_3 \tag{13.a}$$

$$Q_3 = \frac{\partial M_3}{\partial x_3} = -q_1(x_3 - L_3) - q_2 L_3 - P + X'_1 + X'_2 \tag{13.b}$$

The coefficients of flexibilities are obtained through Eq. (6) to (8), due to shear stress and bending moment, applicable in this case. Substituting the problem data in Eq. (11) to (13) and consequentlyin Eq. (6) to (8), we obtain the coefficients of flexibility. Therefore, the equation of compatibility of displacements, Eq. (9), is written in the form:

$$\begin{aligned} -0{,}043106 + 0{,}000566X_1 + 0.001985X_2 &= 0 \\ -0{,}247882 + 0{,}001985X_1 + 0{,}013338X_2 &= 0 \end{aligned} \tag{14}$$

Solving Eqs. (14) simultaneously, results in the statically indeterminate constants: X_1 = 22,98 kN andX_2 = 15,16 kN. Being the reactions in the supports "*B*" and "*C*" respectively, which makes the beam statically determined. Considering only the influence of the bending moment, the equation of compatibility of displacements, Eq. (9), is written in the form;

$$\begin{aligned} -\ 0{,}042255 + 0{,}000540\ X\ 1 + 0.001960\ X\ 2 &= 0 \\ -\ 0{,}246274 + 0{,}001960\ X\ 1 + 0{,}0132608X\ 2 &= 0 \end{aligned} \tag{15}$$

Solving Eq. (15) simultaneously, one has: X_1 = 23,41 kN and X_2 = 15,11 kN. In this case, the shear stress changes by approximately *2%* the final result.

After analyzing the beam by the Ansys® finite element program, considering the BEAM3 elementfor the beam and the COMBIN14 element for the springs, the following results are obtained:X_1 = 23,42 kN, for the vertical displacement δ_1 = 1,17E-3 m and X_2 = 15,11 kN, for the vertical displacement δ_2 = 5,04E-4 m.

Statically indeterminate beam supported on the three springs

Consider the statically indeterminate beam set in support "*A*" and supported on three springs ofelastic constants k_1 = 20000 kN/m, k_2 = 25000 kN/m and k_3 = 30000 kN/m, on supports "*B* ", "*C* " and "*D*" respectively. The beam has a constant bending stiffness *EI* = 18370,8 kNm² and it is subjected to uniformly distributed loads q_1 = 2,0 kN/m and q_2 = 4,0 kN/m, a concentrated load *P* = 10,0 kN and at a concentrated moment *M* = 20,0 kNm and, has dimensions L_1 = 3,0 m, L_2 = 2,0 m and L_3 = 4,0 m, as represented in Figure 3 below.

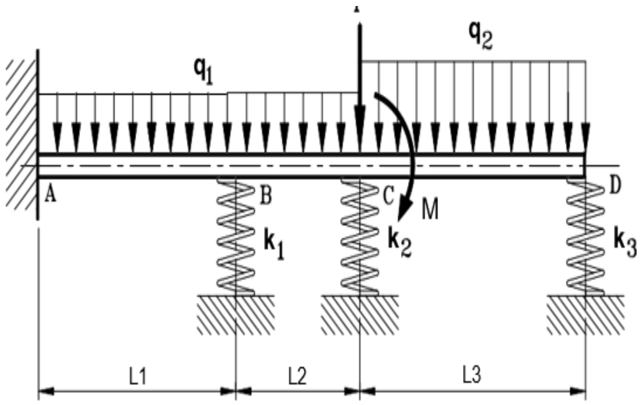

Figure 3 - Beam statically undetermined.

The representation of the adopted cuts, as well as the definition of the statically indeterminateconstants, are represented in Figure 4 below.

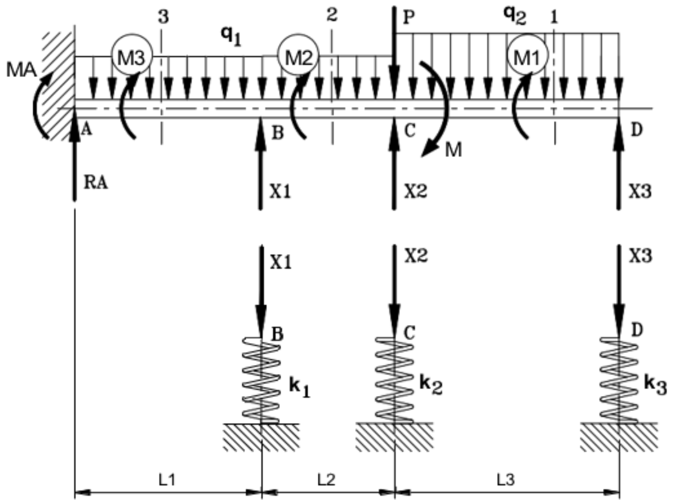

Figure 4: Cuts and statically indeterminate constants.

Based on the verification made in item 3.1, only the influence of the bending moments will be considered, for that, the cuts 1, 2 and 3, defined according to the distribution of loads and supports, are considered. Supports "*B*", "*C*" and "*D*" are defined as superabundant, X_1, X_2 and X_3 being the statically indeterminate constants. In this step, we consider (i = 1, 2 and 3) as virtual loads.

Section 1: $0 \le x1 \le L3$

$$M_1 = -\frac{q_2}{2}x_1^2 + X'_3\, x_1 \tag{16}$$

Section 2: $L_3 \le x_2 \le (L_2+L_3)$

$$M_2 = -\frac{q_1}{2}(x_2 - L_3)^2 - q_2 L_3\left(x_2 - \frac{L_3}{2}\right) - M - P(x_2 - L_3) + X'_2(x_2 - L_3) + X'_3\, x_2 \tag{17}$$

Section 3: $(L_2+L_3) \le x_3 \le (L_1+L_2+L_3)$

$$\begin{aligned} M_3 = &-\frac{q_1}{2}(x_3 - L_3)^2 - q_2 L_3\left(x_3 - \frac{L_3}{2}\right) - M - P(x_3 - L_3) + \\ &+ X'_1\left[x_3 - (L_2 + L_3)\right] + X'_2(x_3 - L_3) + X'_3\, x_3 \end{aligned} \tag{18}$$

Substituting the problem data in Eq. (16) to (18) and consequently in Eq. (6) to (8), we obtain the flexibility coefficients due only to the bending moment. Therefore, the equation of compatibility of displacements, Eq. (9), is written in the form:

$$\begin{aligned} -0{,}042255 + 0{,}000540X_1 + 0{,}000979X_2 + 0{,}001960X_3 &= 0 \\ -0{,}102858 + 0{,}000979X_1 + 0{,}002308X_2 + 0{,}004990X_3 &= 0 \\ -0{,}246274 + 0{,}001960X_1 + 0{,}004990X_2 + 0{,}013261X_3 &= 0 \end{aligned} \tag{19}$$

Solving Eq. (19) simultaneously, one has: X_1 = 1,08 kN, X_2 = 23,06 kN and X_3 = 9,74 kN. Since X_1, X_2 and X_3, statically indeterminate constants, are the reactions in supports "*B*", "*C*" and "*D*" respectively, which makes the beam statically determined.

After analyzing the beam by the Ansys® finite element program, considering the BEAM3 elementfor the beam and the COMBIN14 element for the springs, the following results are obtained:X_1 = 1,08 kN, for the vertical displacement δ_1 = 5,40E-5 m, X_2 = 23,06 kN, for the verticaldisplacement δ_2 = 9,22E-4 m and X_3 = 9,74 kN, for the vertical displacement δ_3 = 3,24E-4 m.

Statically indeterminate frame supported by two springs

Consider the statically indeterminate frame fixed in the support "*A*" and supported of the two springs in the support "*C*". The springs have elastic constants k_1 = 25000 kN/m and k_2 = 40000 kN/m respectively. The frame has a constant bending stiffness EI = 246400 kNm² and, subject to uniformlydistributed loads q_1 = 4,0 kN/m and q_2 = 6,0 kN/m and at a concentrated load P = 10,0 kN. It has dimensions L = 4,0 m, H_1 = 2,0 m and H_2 = 4,0 m, as represented in Figure 5 below.

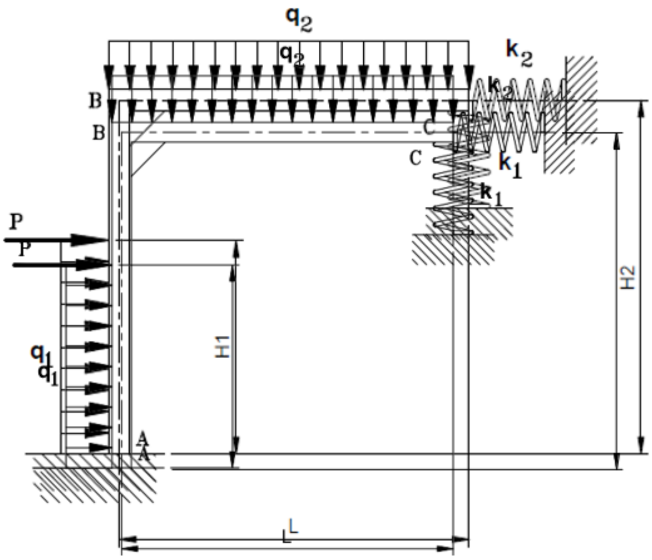

Figure 5: Statically indeterminate frame.

The representation of the adopted cuts, as well as the definition of the statically indeterminateconstants, are represented in Figure 6 below.

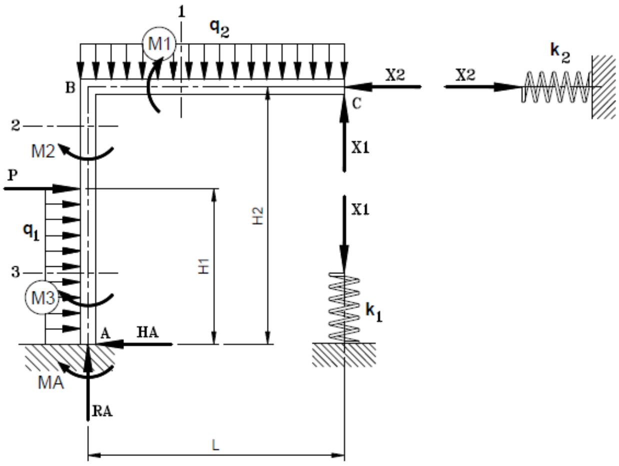

Figure 6: Section and statically indeterminate constants.

For the determination of the bending moments, we consider the sections 1, 2 and 3. The support "C" is defined as superabundant, with X_1 and X_2 being the statically indeterminate constants. In this step, we consider (i = 1 and 2) as virtual loads.

Section 1: $0 \le x_1 \le L$

$$M_1 = -\frac{q_2}{2} x_1^2 + X'_1 x_1 \tag{20}$$

Section 2: $0 \le x_2 \le (H_2 + H_1)$

$$M_2 = -\frac{q_2}{2} L^2 + X'_2 x_2 + X'_1 L \tag{21}$$

Section 3: $(H_2 - H_1) \le x_3 \le H_2$

$$M_3 = -\frac{q_1}{2}\left[x_3 - (H_2 - H_1)\right]^2 - \frac{q_2}{2} L^2 - P\left[x_3 - (H_2 - H_1)\right] + X'_1 L + X'_2 x_3 \tag{22}$$

Substituting the problem data in Eq. (20) to (22) and consequently in Eq. (6) to (8), we obtain the coefficients of flexibility. Therefore, the equation of compatibility of displacements, Eq. (9), is written in the form:

$$\begin{aligned} -0{,}004307 + 0{,}000386 X_1 + 0{,}000130 X_2 &= 0 \\ -0{,}001905 + 0{,}000130 X_1 + 0{,}000112 X_2 &= 0 \end{aligned} \tag{23}$$

Solving Eq. (23) simultaneously, it follows that: X_1 = 8,89 kN and X_2 = 6,72 kN. Since X_1 and X_2 arethe statically indeterminate constants, it makes the frame statically determined.

By analyzing the frame by the Ansys® finite element program, considering the BEAM3 element for the beam and the COMBIN14 element for the springs, the following results are obtained:X_1 = 8,91 kN, for the vertical displacement δ_1 = 3,56E-3 m and X_2 = 6,68 kN, for the vertical displacement δ_2 = 1,67E-3 m.

Statically indeterminate truss supported on the two springs

Consider the statically indeterminate lattice articulated in node "*A*", supported on node "*B*" and supported on two springs, one vertical and one horizontal, in node "*C*". The springs have elastic constants k_1 = 10000 kN/m and k_2 = 20000 kN/m respectively. The bars of the truss have constant axial stiffness EA = 412334,0 kN and it is subjected to a concentrated load P = 10,0 kN inclined from*30°* to the vertical, applied at node "*C*". The trellis has dimensions L = 3,0 m and H = 2,0 m, as shown in Figure 7 below.

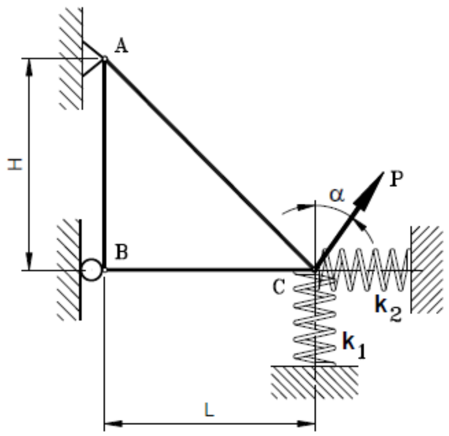

Figure 7: Statically indeterminate truss.

The representation of the bars, as well as of the statically indeterminate constants, are represented in Figure 8 below.

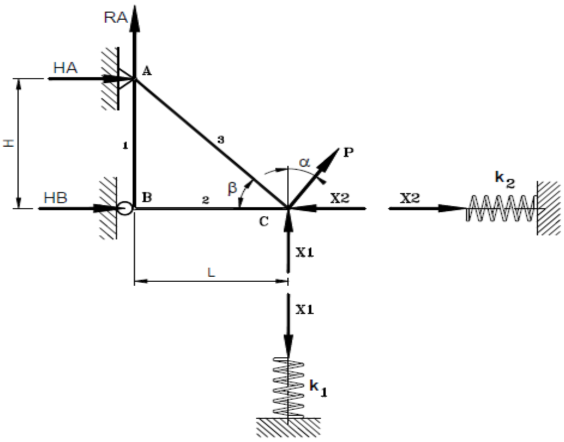

Figure 8: Trellis bars and statically indeterminate constants.

For the determination of the forces acting on the bars, the support "*C*" is defined as superabundant, with X_1 and X_2 being the statically indeterminate constants. In this step, we consider (i = 1 and 2) as virtual loads. The coefficients of flexibilities are obtained through Eq. (6) to (8), due to only the normal internal stress. Since N_n and L_n are constants in each bar, the equations can be presented as follows:

$$\delta_{i0} = \frac{1}{EA}\sum_{1}^{n}\left(N_n L_n \frac{\partial N_n}{\partial X'_i}\right) \tag{24}$$

$$\delta_{ii} = \frac{1}{k_i} + \frac{1}{EA}\sum_{1}^{n}\left[\left(\frac{\partial N_n}{\partial X'_i}\right)^2 L_n\right] \tag{25}$$

$$\delta_{ij} = \frac{1}{EA}\sum_{1}^{n}\left(\frac{\partial N_n}{\partial X'_i}\frac{\partial N_n}{\partial X'_j} L_n\right) \tag{26}$$

The displacement compatibility equation, Eq. (9), is written as:

$$\begin{aligned} &4{,}425.10^{-4} + 1{,}448.10^{-4} X_1 - 1{,}091.10^{-5} X_2 = 0 \\ &-1{,}309.10^{-4} - 1{,}091.10^{-5} X_1 + 5{,}723.10^{-5} X_2 = 0 \end{aligned} \quad (27)$$

Solving Eq. (27) simultaneously, it follows that: X_1 = -2,93 kN and X_2 = 1,73 kN.

Analyzing the lattice by the Ansys® finite element program, considering the LINK1 element for the bars and the COMBIN14 element for the springs, the following results are obtained:X_1 = -2,93 kN, for the vertical displacement δ_1 = 2,93E-4 m and X_2 = 1,73 kN, for the vertical displacement δ_2 = 8,64E-5 m.

Symmetrical parabolic two-hinged arch

Consider the symmetrical parabolic arch fixed in the support's "*A*" and "*B*", the span between the supports is L = 10,0 m, maximum height f = 3,0 m and subjected to a uniformly distributed load$q$ = 6,0 kN/m. The axis of the arc is considered to be a parabola defined by the equation $y = 4fx^2 / L^2$. The cross section is rectangular and constant of width b = 120,0 mm and height h = 300,0 mm,where E = 21,0 GPa.

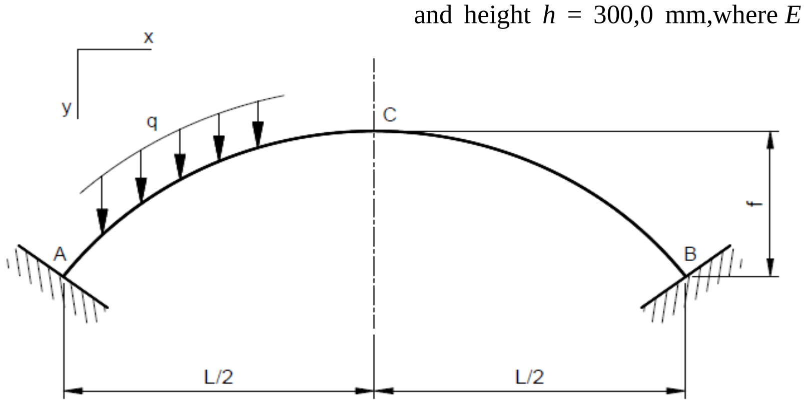

Figure 9: Symmetrical parabolic two-hinged arch.

In this case the "Theorem of Last Work" will be applied for the determination of the statically indeterminate constants. The model for the analysis of the arch with the reactions in the supports, internal stresses and statically indeterminate constants, are represented in figure 10 below.

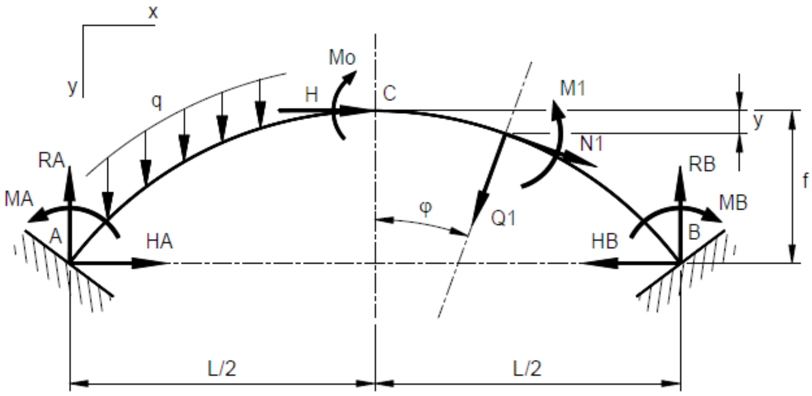

Figure 10: Model for parabolic arch analysis.

From the symmetry condition, from the point "*C*", we have:

$$Tan(\varphi) = \frac{\partial y}{\partial x} = \frac{8fx}{L^2} \quad (28)$$

$$\text{Therefore:} \quad Cos(\varphi) = \sqrt{\frac{1}{1 + \left(\frac{8fx}{L^2}\right)^2}} \quad \text{e} \quad Sen(\varphi) = \sqrt{1 - \frac{1}{1 + \left(\frac{8fx}{L^2}\right)^2}} \quad (29)$$

Transferring the loads from the left half of the arch to the point "*C*", we have:

$$Mo = \frac{qL^2}{8} - MA - Hf \text{, send } RA = \frac{qL}{2} \quad (30a, b)$$

Analyzing the section 1 from the point "*C*", neglecting the effect of the shear forces, one has:

$$M_1 = -\frac{qx^2}{2} + \frac{qL^2}{8} - MA - H(f - y) \tag{31}$$

$$N_1 = -qxSen(\varphi) - HCos(\varphi) \tag{32}$$

From Eq. (10), we have:

$$\delta_{HC} = 0 = \int_0^{\frac{L}{2}} \frac{M_1}{EI}\frac{\partial M_1}{\partial H}dx + \int_0^{\frac{L}{2}} \frac{N_1}{EA}\frac{\partial N_1}{\partial H}dx \tag{33}$$

$$\theta_C = 0 = \int_0^{\frac{L}{2}} \frac{M_1}{EI}\frac{\partial M_1}{\partial MA}dx \tag{34}$$

Solving Eq. (33) and (34), it follows that: H = 24,77 kN and MA = 0,47 kNm.

By analyzing the parabolic arc by the Ansys® [2] finite element program, considering the BEAM3 element for the arc, we obtain the following results: H = 24,66 kN and MA = 0,42 kNm.

Comments and Conclusions

The proposed model based on the classical equations of Timoshenko S.P. and Young D.H. (1965) [1] and the works of Hibbeler R.C. (2013) [3], present good results compared to the numerical results obtained by the Ansys® [2] finite element program. The considerations adopted for disregarding the effects of shear forces did not significantly affect the final results presented. The results presented, based on the classical theories, can help in the analysis of structural elements, being another source of consultation assisting in the definition of the structural parameters and in the boundary conditions.

References

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