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Analytical and Numerical Solutions for the Rocket Equation Considering Air Resistance

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Abstract

We explore solutions to the rocket equation and see that solving the rocket equation with linear drag may be instructive in mathematical technique, but this solution, regardless of frequent appearance in textbooks, is unrealistic for rocket applications. However, we do see that linear drag may be helpful in interpreting the motion of squid or octopus that propel themselves through water in a rocket-like manner. Though analytic solutions to the rocket equation for the quadratic drag case or the mixed quadratic and linear case are more realistic, there is no analytical solution that can include the effects of gravity (and the same is true of linear resistance). A survey given in an intermediate Mechanics class just after conducting this numerical exercise indicates that students understand general concepts related to air resistance, but have more trouble with concepts related precisely to quadratic and linear air resistance. Students perceive that they understand the calculations better than the concepts. Some adjustments to the survey and as well as consistently administering this over time will help us interpret the results more definitively.

Keywords: *air resistance; mechanics; numerical integration; physics pedagogy; rockets.*

I. Introduction

The problem of a rocket experiencing quadratic air resistance is considered, and in particular an analytical rocket equation is derived assuming such a form of air drag. Such a solution, especially when approximating realistic cases, can be very useful pedagogically. They can aid in students' conceptual understanding simultaneous with achieving some mathematical success in reaching a solution. The quadratic air resistance results are compared to rockets experiencing linear air resistance and a form of air resistance including both effects. These results are then compared to numerical solutions using a Euler integration method within Microsoft EXCEL [1]. Finally, the classroom reaction to these exercises is discussed. In the supplementary materials available by contacting the author I include the instructions for creating a template for reproducing the calculations of this article, the resultant template, and a copy of the survey distributed to the students.

Since the inception of space-flight, students have been intrigued by the concepts of rocketry, considering its significance to our future. NASA's return of human space flight to the Moon and possibly beyond, growth in commercial and international spaceflight as well as space tourism have heightened this interest lately. Most mechanics textbooks use concepts of momentum conservation to derive a straightforward rocket equation, usually attributed to Konstantin Tsiolkovsky [2], that expresses final velocity of the rocket in terms of the mass fraction used as fuel. There has been some discussion of the rocket equation and analogous physical situations in both textbooks [3] and in articles in journals such as *The Physics Teacher* [5]. Intermediate mechanics textbooks such as *Introduction to Classical Mechanics [5]* often present somewhat more complex problems with analytical solutions, but often with simplifying assumptions such as dominance of linear air drag, which are only precisely valid for much smaller objects moving at slower speeds, and are thus often unrealistic for rockets. Students can gain understanding by also considering solutions that, though they may seem complicated at first, could be more realistic in addressing the problem at hand [6]. In this case, it would involve introducing quadratic air resistance in place of, or in combination with, linear resistance when considering rocket motion in the atmosphere. Quadratic resistance is a better approximation for fast macroscopic objects. Also, students at this level would have formally studied a number of integration methods in mathematics classes, but may not have seen very many of these methods applied to interesting practical physics problems such as rocketry. Looking at rocket problems more closely gives students this opportunity.

II. Methodology and Results

Before deriving an expression for rocket velocity, it's instructive to consider a simpler case for which a single block of mass M and exhaust velocity v_{ex} with respect to the rocket is ejected out of the back of the rocket of mass m. If we assume that the system starts off with zero velocity and that the final velocity of the rocket is v, then we can determine from the following equation [7]:

$$mv + M(v - v_{ex}) = 0 \tag{1}$$

that the new forward velocity of the rocket is:

$$v = \frac{Mv_{ex}}{m + M} \tag{2}$$

Since a rocket is losing mass continuously and not discretely as above, we now have to consider the conservation of momentum for a rocket of instantaneous mass m while losing mass as exhausted fuel. In this case we can replace the discrete mass, M, with -Δm. The negative sign is needed since we are subtracting from the original rocket mass m. If we consider that the mass is not lost in one chunk, but as a summation of smaller chunks over time, we change to:

$$mv - \sum_{j=1}^{N} \Delta m_j (v - v_{ex}) = 0 \tag{3}$$

And then we can recast Eq. (3) slightly in terms of changes during a short time interval so that we can more easily transition to time derivatives:

$$mv - \sum_{j=1}^{N} \frac{\Delta m_j}{\Delta t_j} \Delta t_j (v - v_{ex}) = 0 \tag{4}$$

But if these changes are infinitesimally small we can use an integration of the continuous of rate of mass loss, $-\dot{m}$, over time in place of the summation. The total ejected fuel mass would be $-\dot{m}(\dot{t})\, dt$ (negative, because of the constant mass loss) and the momentum, considering the relative motion with respect to the rocket, as above, is $-\dot{m}(t)(v - v_{ex})dt$

$$m_0 v_0 = mv - \int_0^{\tau} \dot{m}(v - v_{ex}) dt \tag{5}$$

Here, $m_0$and v_0 refer to (constant and potentially non-zero) initial values. If we take the time derivative of this expression then this becomes:

$$0 = \frac{d}{dt}(mv) - \dot{m}(v - v_{ex}) \tag{6}$$

Combining all $\dot{v}$ and $\dot{m}$ terms, we get:

$$m\dot{v} = -\dot{m} v_{ex} \tag{7}$$

Therefore, as a basis for comparison, we can now derive the traditional ideal rocket equation in an environment free of forces other than the thrust. For such a rocket, the only force involved is the thrust from the ejected exhaust:

$$m\frac{dv}{dt} = Thrust = -v_{ex}\frac{dm}{dt} \tag{8}$$

Where here $\frac{dm}{dt}$ is equivalent to the mass loss rate from the exhaust, $\dot{m}$.

We can integrate both sides over time, which has the effect of eliminating the dt differential on both sides of Eq. (8), and then dividing through by m:

$$dv = -v_{ex}\frac{dm}{m} \tag{9}$$

We can integrate both sides from initial to final values and assuming that the initial velocity is zero:

$$v_f = v_{ex} \ln\left(\frac{m_i}{m_f}\right) \tag{10}$$

Where m_i is the initial mass and is assumed to be 2 x 10^6 kg and m_f is the final mass of 10^5 kg. It is assumed that the exhaust velocity is 3000 meters/sec and that the final mass ratio is a factor of 20. These parameters are chosen to be realistic for modern rockets. For our purposes, v_f can refer to the velocity at any arbitrary point corresponding to a mass ratio from one to 20. We can then enumerate velocities corresponding to any mass ratio in this range. The results are plotted in Fig. 1 as the line with shorter dashes. The greatest velocity (at mass ratio of 20) of approximately 9000 meters/sec is consistent with a rocket speed that is between orbital and escape velocities at the surface of the Earth.

Adding to this, we consider the effects of gravity during a vertical launch. This turns out to lead to a straightforward subtraction of an additional time dependent term [8]:

$$v_f = v_{ex} \ln\left(\frac{m_i}{m_f}\right) - gt \tag{11}$$

Where, in addition to the terms that have their previous definitions, there are also g and t that have their common definitions of the acceleration due to gravity and time, respectively.

Assuming that the mass loss rate due to fuel usage is constant, we can make a substitution for time, t , in the above Eq. (11) using the following formula:

$$t = \frac{m_i - m_f}{k} \tag{12}$$

With otherwise similar assumptions as with the previous case, the results are plotted as the dark solid line in Fig. 1. As expected, after explicitly taking gravity into account, the final velocity will be significantly lower (due to the required work against gravity) than any of the other cases (under similar assumptions of exhaust velocity and initial mass).

We next present analytical solutions for the case of a rocket undergoing air resistance without gravity, which will only be exact in a gravity free environment. With our analytical solutions we can, as before, derive rocket equations solving for final speed of the rocket in terms of the fraction of mass spent as fuel. We also will compare our analytical solutions with a numerical solution determined with a Euler algorithm within EXCEL (both with and without gravity considered).

We can start by writing out Newton's Second Law for the case of a rocket experiencing linear air resistance:

$$m\frac{dv}{dt}\hat{v} = -\dot{m}v_{ex}\hat{v} - bv\hat{v} \tag{13}$$

Here, m refers to the mass of the rocket and fuel, $\hat{v}$ refers to the direction of motion, $\dot{m}$ is the mass loss rate due to exhaust, v_{ex} has its previous definition, b is the linear air drag coefficient and v and t retain their previous meanings. If we then assume that the mass loss is constant, we can replace $-\dot{m}$ with the constant, k. Since we ultimately want a rocket equation that expresses speed in terms of spent mass, we can then transform dt into dm by using the relationship:

$$-\frac{dm}{dt} = k \tag{14}$$

We can then re-arrange by separation of variables, considering an integration of v and t, and assuming no changes in the direction of motion.

We can derive a rocket equation assuming linear air resistance in one dimension [9]:

$$\int_0^{v_f} \frac{k}{kv_{ex} - bv}dv = \int_{m_f}^{m_i} \frac{dm}{m} \tag{15}$$

which leads to the solution:

$$\frac{k}{b}\ln\left(\frac{kv_{ex} - bv_f}{kv_{ex}}\right) = \ln\left(\frac{m_i}{m_f}\right) \tag{16}$$

After both sides are exponentiated, they can be rearranged to solve for velocity in terms of mass ratio and the other rocket and air resistance parameters [10]. A similar form is discussed in Gregory's *Classical Mechanics [11]*:

$$v_f = \frac{k}{b}v_{ex}\left[1 - \left(\frac{m_f}{m_i}\right)^{\frac{b}{k}}\right] \tag{17}$$

For our next set of calculations based on Eq. (17), b is assumed to be 1 x10^3, and we assume an exhaust velocity, v_{ex} of about 3000 meters/sec and k of 1 x 10^4 kg/sec. This value assumed for b (depends on the linear size of the object and density of the medium), though it leads to reasonable final velocity values (see Fig. 1), is particularly unrealistic, with b many orders of magnitudes larger than what is typical for air at STP. This assumed value of b might be reasonable for a rocket going through a highly viscous fluid which, though conceptually possible, is beyond most typical applications. However, some investigators have mentioned that linear drag may be relevant for water rockets used for classroom experiments due to their relatively slow speeds [12]. A possible example from nature might be the locomotion of cephalopods such as squid and octopus [13]. They siphon seawater, then expel it at high speeds in order to move. If we assume b of 1.3 x 10^{-2}, v_{ex} of 45 m/sec, k=0.1 kg/sec, and a mass ratio of about 0.94, then the final velocity is about 2.8 meters/sec which is roughly consistent with known speeds of such marine animals (the final velocity is not highly dependent on the parameter assumptions). We will later examine this example considering quadratic drag or a combined drag model. Another alternative for analyzing the rocket example, say, adopting reasonable values of b for air yet also adopting much lower values of the mass loss rate, k , is infeasible, at least at launch from Earth, as it suggests that the rocket will take years to get to the target velocity. This might be plausible for accelerating over large distances in space, though with b very small. Equation (17) is plotted as the dotted line in Fig. 1. It should be noted that solutions will scale with $\frac{k}{b}$, that is if k and b increase or decrease in the same way between two solutions, the plotted results will be identical provided that the exhaust velocity and mass ratios are similar. However, we should caution that this would mathematically allow for potentially unphysical values of k or b, and we would not be able to differentiate between the two from an isolated plot. However, it is still useful to plot the results so long as we cautious in noting the range of constants k and b.

We can repeat this process for quadratic air resistance, following from Eq. (13), but with the drag force replaced by $-cv^2$ where $c \equiv \frac{1}{2}c_D\rho A$ with c_D referring to the quadratic air drag coefficient, ρ refers to the density of the medium (air density in this case) and A is the cross-sectional area of the object moving through the medium.

$$m\frac{dv}{dt}\hat{v} = -\dot{m}v_{ex}\hat{v} - cv^2\hat{v} \tag{18}$$

We then set up the integration using the separation of variables technique we used previously:

$$\int_0^{v_f} \frac{1}{\frac{k}{m}v_{ex} - \frac{c}{m}v^2} dv = \int_0^t dt' \tag{19}$$

Then similarly to Eq. (15), we can then express the velocity integration as:

$$\int_0^{v_f} \frac{k}{kv_{ex} - cv^2} dv = \int_{m_f}^{m_i} \frac{dm}{m} \tag{20}$$

This leads to an arctanh solution for the left-hand side and a natural logarithm solution for the right-hand side, which can then be inverted to get the following:

$$v_f = \sqrt{\frac{kv_{ex}}{c}} \tanh\left[\frac{v_{ex}}{\sqrt{\frac{kv_{ex}}{c}}} \ln\left(\frac{m_i}{m_f}\right)\right] \tag{21}$$

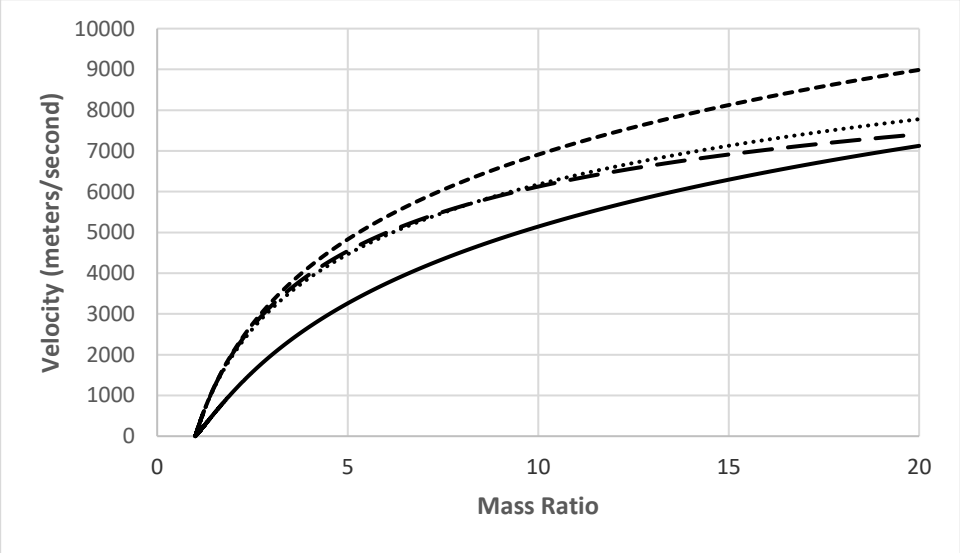

Figure 1: Velocity versus mass ratio. The exhaust velocity, v_{ex} is assumed to be 3000 meters/sec. The line at the top with the shorter dashes represents the rocket free of external forces. The line with the smaller dots represents the case that includes linear air resistance, but not gravity (see text for other important assumptions). The longer dashed line which is almost overlaying the previously mentioned dotted line includes the effects of quadratic air resistance, but not gravity, and the dark solid line on the bottom includes gravity as the only external force. These are all calculated with the analytic formulae given above.

For quadratic air resistance, we assume in our calculations for the line with longer dashes in Fig. 1 (using Eq. (21)), $C_D = 0.17$, ρ_{air}=4.6 x 10^{-2} kg/m^3, and A=63.8 m^2. The values for these parameters are representative for a typical rocket at some point moving through the Earth's atmosphere. A more detailed model could track the changes of all of these parameters with atmospheric altitude, but is beyond the scope of the present work. We can also reconsider the case of the cephalopod, this time with quadratic drag parameters. We can use the same exhaust velocity (45 m/sec) and k (0.1 kg/sec) but now use c of approximately 0.4, which would be appropriate for a squid-sized object in water. Using Eq. (21) we calculate a final speed of about 2.3 meters/sec, which is similar to results using the previous linear drag parameters).

Speaking to why some textbooks seem to focus on linear drag solutions even in situations where they are clearly unrealistic, it's possible that they favor presenting the linear drag solution over the quadratic solution due to many students' lack of familiarity with hyperbolic trigonometry. However, texts such as John Taylor's *Introduction to Classical Mechanics* had already brought up such forms in earlier chapters [14] in a very similar solution for the case of a falling ball.

Next, we show the results of a numerical integration using a Euler algorithm within EXCEL. We can numerically integrate the resulting equation to determine velocity by breaking up the calculation into N cells and then in each individual cell, calculate the velocity by multiplying the acceleration by the time interval between cells, Δt, and adding this to the velocity from the previous cell. The velocity in any cell is then [15]:

$$v_n = a_{n-1}\Delta t + v_{n-1}\,. \tag{22}$$

This is essentially equivalent to Euler's method for evaluating first order differential equations for which the solution for the differential equation given above for Newton's 2nd Law becomes

$$v(t+\Delta t) = v(t) + \Delta t \frac{dv}{dt}. \tag{23}$$

Here the higher order Euler terms involving second and third derivatives have been dropped [16].

If the force of gravity is excluded and the same assumptions are made as directly above, the results of the numerical integration are similar to quadratic resistance analytical results we have previously calculated, and can be seen as the dashed line in Fig. 2 and the line with the longer dashes in Fig. 1. However, after adding the effects of gravity, we can see (solid dark line in Fig. 2) that for similar choices of parameters, the final velocity will be approximately 15% smaller. To get a similar final velocity for the case with gravity (Fig. 2) as for the case without gravity in Fig. 1, the product of ρ, A, c_D would have to be smaller by greater than a factor of two. Since the geometry dependent rocket parameters (c_D and A) would be the same for each of these cases, this would suggest assuming lower density of the medium when considering the effects of gravity.

Additionally, we can consider a case for which both forms of air resistance need to be considered simultaneously. For this case we can start with force equation much like Eq. (13):

$$m\frac{dv}{dt}\hat{v} = -\dot{m}v_{ex}\hat{v} - bv\hat{v} - cv^2\hat{v} \tag{24}$$

This will then lead to an integration similar to Eq. (15) and Eq. (20):

$$\int_0^{v_f} \frac{k}{kv_{ex} - bv - cv^2}\, dv = \int_{m_f}^{m_i} \frac{dm}{m} \tag{25}$$

Following integral tables [17], the form of the solution of the integral on the left-hand side is:

$$\frac{k}{\sqrt{b^2+4ckv_{ex}}} \ln\left[\frac{-2cv - b - \sqrt{b^2+4ckv_{ex}}}{-2cv - b + \sqrt{b^2+4ckv_{ex}}}\right] \tag{26}$$

Which then, after evaluation at the limits and equating with the right-hand side of Eq. (25) leads to:

$$v_f = \frac{\left(b+\sqrt{b^2+4ckv_{ex}}\right)\left[\left(\frac{m_i}{m_f}\right)^{\frac{\sqrt{b^2+4ckv_{ex}}}{k}} - 1\right]}{2c\left[1 - \left(\frac{b+\sqrt{b^2+4ckv_{ex}}}{b-\sqrt{b^2+4ckv_{ex}}}\right)\left(\frac{m_i}{m_f}\right)^{\frac{\sqrt{b^2+4ckv_{ex}}}{k}}\right]} \tag{27}$$

Assuming similar values of the constants (ρ,c_D ,A) though now using a much lower and realistic value of b leads to results that are essentially identical to the case with just quadratic resistance included in Fig. 2.

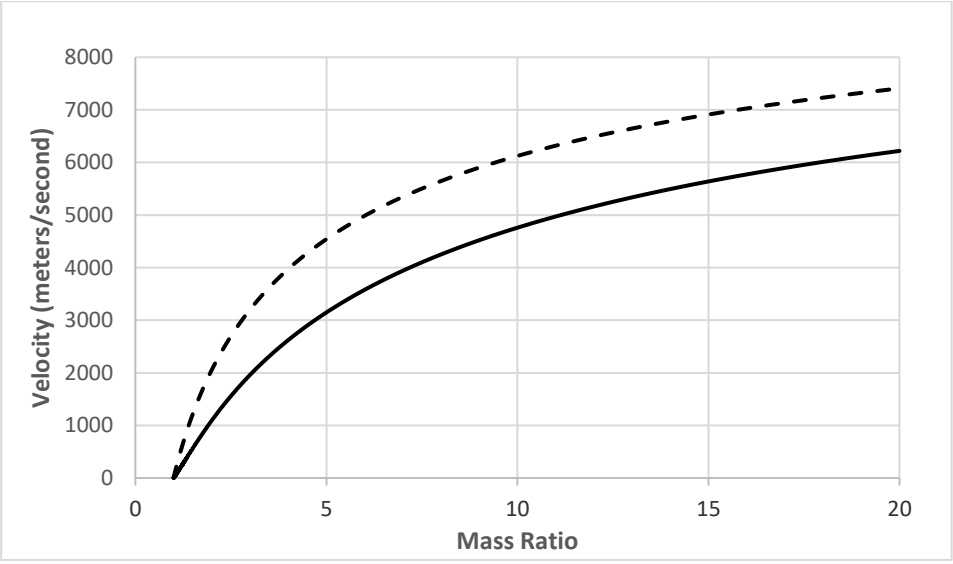

Figure 2: Velocity vs. Mass ratio using a Euler algorithm within EXCEL. Exhaust velocity, v_{ex}, is assumed to be 3000 meters/sec. The solid line includes the effects of gravity and quadratic resistance and is the result of an EXCEL numerical calculation. The dashed line is the EXCEL numerical calculation assuming quadratic drag and excludes the effects of gravity. On this scale, the analytic calculation that excludes gravity would overlap the plotted dashed curve).

Considering Eq. (27), linear drag effect will appreciably add to the quadratic drag effect if b isn't too small as compared to $\sqrt{4ckv_{ex}}$. Generally, this won't be the case because even in going from air to water, the b term just isn't large enough.

III. Classroom Reaction

In an intermediate Mechanics class, we have introduced an exercise using EXCEL that numerically calculates the speed of a rocket as a function of mass ratio including the effects of gravity, thrust and drag forces. As part of the exercise, students are asked to compare their numerical results with the analytical solutions. After this exercise was completed we administered a 20-question survey (also included in the supplementary materials). The first half of the survey focused on conceptual questions related to air drag and rockets. The second part of the survey was comprised of 10 Likert-scale questions gauging students' perceptions of their own learning of certain concepts. Of the conceptual questions with the highest percentage answered correctly, one was Q3 (given that Mars has lower mass and less dense atmosphere than Earth, the question has students pick which pair of forces will be greater or lesser than on Earth). They all correctly picked that both gravity and drag would be lower. Another question (Q4) asked which forces for a vertically launched rocket would be acting in the same direction (they correctly picked drag would be acting in the direction opposite to thrust). Those with the lowest percentage answered correctly were related to linear drag. One such question concerned a concept from the science fiction film, *Fantastic Voyage*. In that film, a submarine is shrunk to microscopic size so that it can perform life-saving medical procedures on a person from inside them. It was asked whether this submarine at this size, acting in the blood stream, would experience linear or quadratic drag. The intended answer was that its linear drag because the object is now small enough that its interacting with individual molecules. It's quite possible that several aspects of this question confused the matter, such as lack of familiarity with the film, and the conditions involved being rather fantastical. Thus, it's very likely that this question will be replaced with something a bit more direct (but still testing similar linear drag concepts) in future surveys.

Of the Likert-scale questions, the highest level of agreement was on questions 11 (score of 4.43) and 19 (score of 4) which stated "I understand Newton's Second Law" and "The numerical exercises help me understand problem sets." This is encouraging, as it shows that students are both confident in their overall understanding of concepts and also feel that numerical exercises are of help. However, the lowest Likert-scale scores on questions 14 (score of 2.43) and 16 (score 2.57), "I understand drag calculations, but not the concepts" and "I understand the rocket calculations, but not the concepts" show that there is more work to do in rounding out student knowledge and confidence. For these last two questions, the oppositely phrased questions were also asked. As expected, the Likert score was much higher (score of 3.57) for "I understand drag concepts, but not the calculations;" however, for "I understand the rocket concepts, but not the calculations," the Likert score was just about as low (score of 2.74), suggesting that students were not particularly confident regarding either their understandings of the concepts or the calculations.

In the future, it may help to get a baseline survey towards the early part of the semester and compare the results of the survey after performing this exercise to the results obtained earlier.

IV. Conclusions

We can conclude from these results that solving for a linear drag rocket equation may be instructive in mathematical technique, and though this solution is realistic for describing the motion of cephalopods, it's unrealistic for rockets. Though analytic rocket equations for the quadratic drag case or the mixed quadratic and linear case are more realistic, there is unfortunately no analytical solution that can include the effects of gravity (and the same is true of linear resistance). Numerical analysis can be used to accurately estimate the effects of gravity and allow us to estimate the accuracy of the analytical solutions presented above. Comparing these cases can lead students to better intuitively interpret Newton's Law 2nd Law while also performing straightforward calculations. A survey given just after this numerical exercise indicates that students understand general concepts related to air resistance, but have more trouble with concepts related precisely to quadratic and linear air resistance. Also, students perceive that they understand the calculations better than the concepts. Some adjustments to the survey and as well as consistently administering this over time will help us interpret the results more definitively.

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